Groups Simplified

Let us have a set G together with a binary operation $*$. We will use multiplicative notation throughout meaning $ab = a * b$. Let $x, y, z \in G$. If $\langle G, * \rangle$ has the following properties:

1. $(xy)z = x(yz)$
2. $ex = x$
3. $x^{-1}x = e$

for some fixed $e \in G$, then we say that $\langle G, * \rangle$ is a group. In class, we needed to show that $xe = x$ and $xx^{-1} = e$. However, these can be derived by the prior properties.

Prove $xx^{-1} = e$

$$
\begin{aligned}
e &= (xx^{-1})^{-1}(xx^{-1}) \\
&= (xx^{-1})^{-1}(x(ex^{-1})) \\
&= (xx^{-1})^{-1}(x((x^{-1}x)x^{-1})) \text{ —— (A)} \\
&= (xx^{-1})^{-1}(x(x^{-1}x)x^{-1}) \\
&= (xx^{-1})^{-1}((xx^{-1})xx^{-1}) \\
&= (xx^{-1})^{-1}((xx^{-1})(xx^{-1})) \\
&= ((xx^{-1})^{-1}(xx^{-1}))(xx^{-1}) \\
&= e(xx^{-1}) \\
&= xx^{-1}
\end{aligned}
$$

Prove $xe = x$

Once we showed $xx^{-1} = e$, the proof of $xe = e$ is simple.

$$
\begin{aligned}
x &= ex \\
&= (xx^{-1})x \\
&= x(x^{-1}x) \\
&= xe
\end{aligned}
$$